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EXPLORING THE SPACE OF COMPACT SYMMETRIC CMC SURFACES

LYNN HELLER, SEBASTIAN HELLER, AND NICHOLAS SCHMITT

Abstract. We map out the moduli space of Lawson symmetric constant mean curvature surfaces in the 3-sphere of genus $g > 1$ by flowing numerically from Delaunay tori with even lobe count via the generalized Whitham flow.

1. **Overview.** In this note we map out a portion of the moduli space of embedded constant mean curvature (CMC) surfaces in the 3-sphere experimentally by a numerical implementation of the generalized Whitham flow [2]. This provides numerical evidence for the existence of the flow reaching arbitrary genus.

Experiment. *For each pair of integers $g \geq 1$ and $n \geq 0$ we construct numerically a 1-parameter family Ξ_g^n of compact Alexandrov embedded CMC surfaces of genus g in $\mathbb{S}^3$, with n controlling the lobe count:*

- *the family Ξ_g^0 starts at the Lawson surface $\xi_{g,1}$ and converges to a chain of two minimal spheres;*
- *the family Ξ_g^n ($n \geq 1$) converges at one end to a chain of $(g+1)n$ CMC spheres and at the other to a chain of $(g+1)n+2$ CMC spheres.*

Each surface in Ξ_g^n has a cyclic symmetry of order $g+1$ with four fixed points.

These Ξ_g^n families were computed numerically via the generalized Whitham flow [2], a topology-breaking flow through CMC surfaces in $\mathbb{S}^3$ which starts at CMC tori and, as indicated by numerical evidence, reaches closed CMC surfaces of arbitrary genus.

The generalized Whitham flow passes through each of the families Ξ_g^n with n fixed and g increasing arbitrarily, starting at the tori Ξ_1^n. To describe this initial data, recall [4] the embedded CMC tori in the 3-sphere consist of the 1-parameter family of homogeneous tori of increasing mean curvature starting at the minimal Clifford torus, along which bifurcate 1-parameter families of n-lobed Delaunay (equivariant) tori at sequencial bifurcation points β_m (see figure 2). The initial family Ξ_1^0 is made up of the homogenous tori between the Clifford torus and β_2, together with the 2-lobed Delaunay tori. The initial family Ξ_1^n ($n > 0$) is made up of the $(2n)$-lobed Delaunay tori, the homogeneous tori between β_{2n} and β_{2n+2}, and the $(2n+2)$-lobed Delaunay tori.

The topology-breaking flow is described qualitatively as follows. The initial torus in Ξ_1^n has a cyclic symmetry of order two with four fixed points. The flow preserves the topology of the torus minus two disks, formed by introducing two cuts connecting the fixed points in pairs. The flow retains the rotational symmetry, decreasing its angle α from π to 0. When $\alpha = 2\pi/(g+1)$, $g \in \mathbb{N}$, the surface can be completed by the rotational symmetry to a closed compact unbranched surface of genus g. At other angles $\alpha \in 2\pi\mathbb{Q}$, the surface can be completed to a closed surface branched at four points.

The generalized Whitham flow starting at a $(2n)$-lobed Delaunay torus can flow either to Ξ_g^{n-1} or Ξ_g^n. This flow direction is determined by the choice of order two symmetry: the symmetry with axes through necks flows to Ξ_g^{n-1} while the symmetry with axes through bulges flows to Ξ_g^n (see figure 3).

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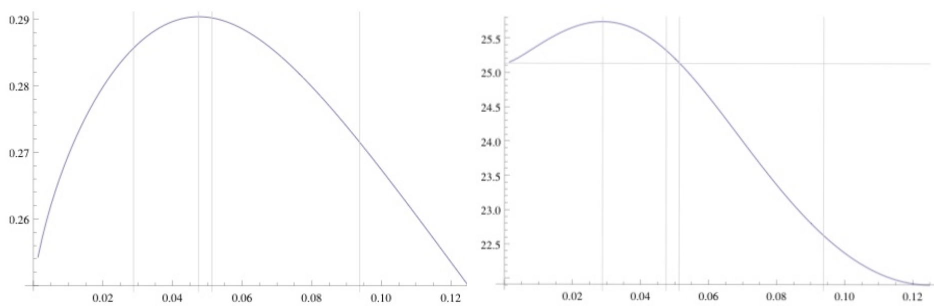

FIGURE 1. The above plots show the mean curvature (left) and Willmore energy (right) along the family of genus 2 CMC surfaces Ξ_g^0 depending on its conformal type encoded as the poleangle $\theta \in [0, 1/8]$. For each plot the family Ξ_g^0 starts at the Lawson surface (right) and ends at the chain of two spheres (left): the four poles of the DPW potential (umbilics of the surface) are $\pm e^{\pm 2\pi i\theta}$. The events from right to left are marked by vertical lines: 1. Branchpoint on unit circle, 2. Willmore energy $= 8\pi$, 3. Maximum mean curvature, 4. Maximum Willmore energy ($> 8\pi$).

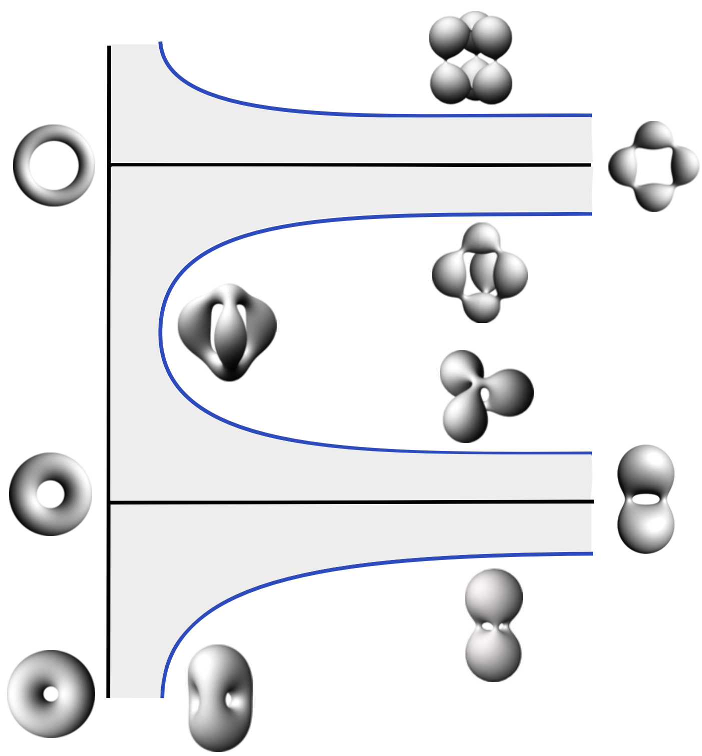

FIGURE 2. The moduli space of Lawson symmetric CMC surfaces in $\mathbb{S}^3$ arising from even-lobed Delaunay tori. The vertical line represents homogeneous tori of increasing mean curvature starting at the minimal Clifford torus at the bottom. The horizontal lines represent Delaunay tori with increasing even number of lobes (bottom to top). The curved lines represent the 1-parameter families Ξ_2^n of genus 2 surfaces. The flow occurs in the shaded regions; The families Ξ_g^n of genus $g > 2$ (not shown) are obtained by continuing the shaded regions beyond the curved lines.

The families Ξ_g^0 and Ξ_g^1 were first discovered in [3] by numerical search. We have numerically computed the mean curvature and the Willmore energy of Ξ_g^0, see Figure 1.

2. **The potential.** We construct the Ξ_g^n families numerically via the generalized Weierstrass representation [1] for CMC surface in $\mathbb{S}^3$. The Weierstrass data consists in a $\mathrm{sl}_2\mathbb{C}$ loop-valued potential ξ with appropriate asymptotics in the spectral parameter λ. The CMC immersion is obtained as $F(\lambda_0)F^{-1}(\lambda_1)$, where $\lambda_0,\, \lambda_1 \in \mathbb{S}^1_\lambda$ are the *sym points*, and F is the unitary factor of the loop group Iwasawa factorization of Φ solving the ODE $\mathrm{d}\Phi = \Phi\xi$.

The potential ξ for the Ξ_g^n families is a Fuchsian potential on $\mathbb{CP}^1$ with four simple poles

$$\xi := \sum_{k=0}^{3} \frac{A_k \mathrm{d}z}{z - z_k} \tag{0.1}$$

with order 2 symmetry $\delta^*\xi = \sigma^{-1}\xi\sigma$, $\delta(z) := -z$, $\sigma := \mathrm{diag}(i,\, -i)$ and real symmetry $\xi(\overline{z},\, \overline{\lambda}) = \overline{\xi}$.

The asymptotics of ξ in the spectral parameter λ is determined by the requirements of the generalized Weierstrass representation and the Hopf differential of the surface: the upper right entry of the residue A_0 has a simple pole at $\lambda = 0$, the lower left entry of A_0 has a simple zero at $\lambda = 0$, and the residues of ξ have no other poles in the unit disk in $\mathbb{C}_\lambda$.

To reach the Ξ_g^n families, we impose the condition that the eigenvalues $\pm\nu_0$, $\pm\nu_1$, be real and λ-independent, with $\nu_0 \in (0,\, 1/4]$, $\nu_1 \in [1/4, /1)$, $\nu_0 + \nu_1 = 1/2$. This condition arises due to the fact that the eigenvalues control the angle α of the rotational symmetry being opened. With these assumptions, the potential ξ is a simpler replacement for the potential described in equation 2.1 in [3] , to which it is gauge equivalent.

We note that the points in the punctured unit disk in $\mathbb{C}_\lambda$ at which the parabolic structure corresponding to ξ is unstable are those points for which the two eigenlines of A_1 and A_2 corresponding to the positive eigenvalues coincide. We observed that the family Ξ_g^n, $g > 1$ has n unstable points in the unit disk.

2.1. *Geometric parameters.* The data determining a CMC surface in $\mathbb{S}^3$ via the generalized Weierstrass representation is its potential ξ, two sym points in $\mathbb{S}^1_\lambda$, and the initial condition for the ODE $\mathrm{d}\Phi = \Phi\xi$. For Ξ_g^n families, this data consists of three geometric parameters together with accessory parameters (coefficients of the residues of ξ). The initial condition for the ODE $\mathrm{d}\Phi = \Phi\xi$ is determined as the diagonal unitarizer of the monodromy, unique up to isometry of $\mathbb{S}^3$.

The three real geometric parameters $(\gamma,\, \alpha,\, H)$ are

- the angle $\alpha := 4\pi\nu_0$ of the rotational symmetry being opened;
- the conformal type $\gamma := [z_0,\, -z_1,\, -z_0,\, z_1] \in \mathbb{R}$ of the four punctured $\mathbb{CP}^1$,
- the mean curvature $H := i(\lambda_0 + \lambda_1)/(\lambda_0 - \lambda_1)$.

2.2. *Accessory parameters.* The eigenvalues of the residues of ξ, determining the angle of the rotational symmetry, must be controlled during the flow. Since the complex dimension of the space of monodromy representations on the 4-punctured sphere with fixed eigenvalues is, roughly speaking, 2, the residues of ξ can be parametrized by two meromorphic functions $\hat{x}$ and $\hat{y}$ of λ. For numerical calculations, we represent $\hat{x}$ and $\hat{y}$ as truncated power series in λ at $\lambda = 0$. Because the monodromy of ξ is to be evaluated on the unit circle $\mathbb{S}^1_\lambda$, we require that the potential ξ is holomorphic in the punctured unit disk. This holomorphicity is achieved by the introduction of polynomials and constraints on these polynomials.

More precisely, let $\hat{x}$, $\hat{y}$ be functions on the unit disk with $\hat{x}$, $1/\hat{x}$ and $\hat{y}$ holomorphic. Let p_k, q_k, $(k \in \{0, \dots, 3\})$ be polynomials satisfying the constraints that p_k monic and

$$e_{jk} := (p_j q_j - \nu_j) - (p_k q_k - \nu_k)\ , \quad (j,\, k \in \{0, \dots, 3\}) \tag{0.2}$$

vanishes. Under this constraint, the residues of the potential ξ can be parametrized by p_k, $q_k, \hat{x}$, $\hat{y}$, $(k \in \{0, \dots, 3\})$ as

$$A_0 = \begin{bmatrix} -y & p_0 p_2/\lambda \\ -y_0 y_2 \lambda & y \end{bmatrix}\ , \quad A_1 = \begin{bmatrix} y & -y_1 y_3/\hat{x} \\ p_1 p_3 \hat{x} & y \end{bmatrix} \tag{0.3}$$

$$p := p_0 p_1 p_2 p_3\ , \quad y_k := q_k + p\hat{y}/p_k\ , \quad y := \nu_k + p_k y_k\ , \tag{0.4}$$

where $\nu_2 := -\nu_0$ and $\nu_3 := -\nu_1$.

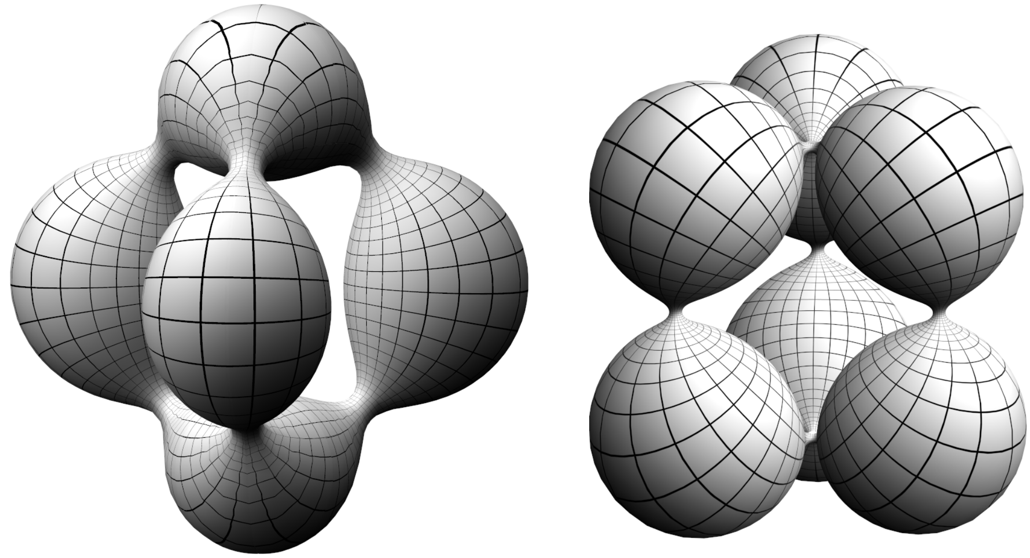

FIGURE 3. Genus 2 surface in Ξ_2^1 with five lobes (left). Genus 2 surface in Ξ_2^2 with six lobes (right).

For numerical computation, the accessory parameters are

$$A := (\operatorname{coeff} p_0, \dots \operatorname{coeff} p_3 \,|\, \operatorname{coeff} q_0, \dots, \operatorname{coeff} q_3 \,|\, \hat{x}_0, \dots, \hat{x}_N \,|\, \hat{y}_0, \dots, \hat{y}_N) \tag{0.5}$$

where coeff q denotes the coefficients of a polynomial q, and the series

$$\hat{x} = \sum_{k=0}^{\infty} \hat{x}_k \lambda^k \ , \quad \hat{y} = \sum_{k=0}^{\infty} \hat{y}_k \lambda^k \tag{0.6}$$

are truncated to power N. The constraints (0.2) are

$$C_A := (\operatorname{coeff} e_{01},\, \operatorname{coeff} e_{02},\, \operatorname{coeff} e_{03}) \ . \tag{0.7}$$

3. **The flow.** The generalized Whitham flow is defined to preserve intrinsic and extrinsic closing conditions. This flow is an implicit infinite dimensional ODE computed numerically by truncation to a finite implicit system of the form $A\dot{X} + B = 0$; $\dot{X}$ is obtained as the least squares solution to this system. The coefficients of the system depend on the monodromy of the potential ξ, computed by a separate nested ODE.

3.1. *Intrinsic closing condition.* The intrinsic closing condition is that the monodromy of ξ is unitarizable along the unit circle $\mathbb{S}^1_\lambda$. More concretely, let M_k $(k \in \{0, \dots, 3\})$ be the monodromy generators for ξ along a curve based at $z = 0$ which winds once counterclockwise around the pole z_k, and let $t_{jk} := \frac{1}{2} \operatorname{tr} M_j M_k$. By proposition 2 in [3], the monodromy is unitarizable when $t_{ij} \in (-1, 1)$. Hence we impose the constraint along $\mathbb{S}^1_\lambda$

$$c_I := (\operatorname{Im} t_{01} \ , \operatorname{Im} t_{02} \ , \operatorname{Im} t_{03} \ , \operatorname{Im} t_{12} \ , \operatorname{Im} t_{13} \ , \operatorname{Im} t_{23}) \ . \tag{0.8}$$

For numerical computation, this constraint is implemented by imposing the condition (0.8) at S equidistant sample points $\mu_k = e^{2\pi i k/S}$, $(k \in \{0, \dots S-1\})$ along $\mathbb{S}^1$. For the flow, the number S of sample points must be large relative to the number N. This is the vanishing of

$$C_I := (c_I(\mu_0), \, \dots, c_I(\mu_{S-1})) \ . \tag{0.9}$$

3.2. *Extrinsic closing condition.* The extrinsic closing conditions are that every monodromy of the unitary frame M satisfies $M(\lambda_0) = M(\lambda_1) \in \{\pm \mathbb{1}\}$ at the two sympoints $\lambda_0,\, \lambda_1 \in \mathbb{S}^1$. By proposition 1 in [3], this is the condition that

$$c_E := [\ell_0,\, \ell_1,\, \ell_2,\, \ell_3] - [z_0,\, -z_1,\, -z_0,\, z_1] \tag{0.10}$$

vanishes to first order in λ at each of the two sympoints, where $\ell_k \in \mathbb{C}\mathrm{P}^1$ $(k \in \{0, \ldots, 3\})$ is the eigenline of A_k corresponding to its positive eigenvalue. With prime denoting the derivative with respect to λ, this is the vanishing of

$$C_E := \big(c_E(\lambda_0),\, c_E'(\lambda_0),\, c_E(\lambda_1),\, c_E'(\lambda_1)\big)\ . \tag{0.11}$$

3.3. *The flow.* The flow is defined in terms of the function f, defined to vanish when the intrinsic and extrinsic closing conditions are satisfied:

$$\begin{bmatrix}\text{geometric parameters } (\gamma,\, \alpha,\, H)\\ \text{accessory parameters } A\end{bmatrix} \overset{f}{\longmapsto} \begin{bmatrix}\text{intrinsic closing condition } C_I\\ \text{extrinsic closing condition } C_E\\ \text{accessory parameter constraint } C_A\end{bmatrix}\ . \tag{0.12}$$

To flow along a curve in the 2-dimensional isosurface $f^{-1}(0)$ we consider

$$t \overset{Y}{\longmapsto} (t,\, u,\, A) \overset{h}{\longmapsto} (\gamma,\, \alpha,\, H,\, A) \overset{f}{\longmapsto} (C_I,\, C_E,\, C_A)\ , \tag{0.13}$$

where t is the real flow parameter, u is a real free parameter, h is an explicit immersion controlling the direction and speed of the flow, and Y is the sought function defined implicitly by the condition $F \circ Y = 0$, where $F := f \circ h$. With dot denoting the derivative with respect to t, Y is defined by the implicit ODE

$$\mathrm{d}F\,\dot{Y} = 0\ . \tag{0.14}$$

In matrix form,

$$\begin{bmatrix}B & A\end{bmatrix}\begin{bmatrix}1\\ \dot{X}\end{bmatrix} = 0\ , \quad \text{that is} \quad A\dot{X} + B = 0\ . \tag{0.15}$$

The vector field $\dot{X}$ is obtained as the least squares solution to the system $A\dot{X} + B = 0$.

The generalized Whitham flow starts with the initial data for a homogeneous or Delaunay torus [2], with $(\gamma,\, \alpha,\, H) = (\text{constant},\, t,\, u)$, reaching a surface in Ξ_g^n by increasing genus. Starting from such a surface, the Whitham flow moves along the Ξ_g^n family, with $(\gamma,\, \alpha,\, H) = (t,\, \text{constant},\, u)$.

4. **Lawson symmetric surfaces.** A Lawson symmetric CMC surface is a compact CMC surface in $\mathbb{S}^3$ of genus $g \geq 1$ which enjoys a cyclic symmetry of order $g+1$ with four fixed points. The families Ξ_g^n described above are Lawson symmetric, and have an additional symmetry induced by the hyperelliptic involution. We conjecture, for each pair of integers $g \geq 1$ and $n \geq 1$, the existence of an additional 1-parameter family $\widehat{\Xi}_g^n$ of Alexandrov embedded Lawson symmetric surfaces which lack the symmetry induced by the hyperelliptic involution. This family is reachable via the generalized Whitham flow from the $(2n+1)$-lobed Delaunay tori, and converges to a chain of $(g+1)n+1$ CMC spheres.

Conjecture. *The space of Alexandrov embedded Lawson symmetric CMC surfaces consists of the families Ξ_g^n and $\widehat{\Xi}_g^n$.*

In the case $g = 1$, this moduli space is connected. In the case of fixed $g > 1$, the families Ξ_g^n ranging over $n \in \mathbb{N}$ are disconnected from each other.

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Lynn Heller, Institut für Mathematik, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tüubingen, Germany

E-mail address: lynn-jing.heller@uni-tuebingen.de

Sebastian Heller, Institut für Mathematik, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tüubingen, Germany

E-mail address: heller@mathematik.uni-tuebingen.de

Nicholas Schmitt, Institut für Mathematik, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany

E-mail address: nschmitt@mathematik.uni-tuebingen.de